IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Wigner-Eckart theorem for the quantum group $U_q(n)$

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1992 J. Phys. A: Math. Gen. 25 2919

(http://iopscience.iop.org/0305-4470/25/10/020)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.58
The article was downloaded on 01/06/2010 at 16:30

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 25 (1992) 2919-2927. Printed in the UK

Wigner–Eckart theorem for the quantum group $U_q(n)$

A U Klimyk

Institute for Theoretical Physics, Kiev 130, Ukraine

Received 28 October 1991

Abstract. Tensor operators transforming under finite dimensional irreducible representations of the quantum group $U_q(n)$ are defined. Using them tensor operators transforming under representations of the quantum algebra $U_q(u(n))$ are introduced. The Wigner–Eckart theorem on matrix elements of tensor operators defined is derived.

1. Introduction

The Wigner–Eckart theorem is one of the most fundamental mathematical results in the theory of symmetries. In the last decade new mathematical objects of the theory of symmetries appeared. They are quantum groups and algebras. In order to apply representations of quantum groups and algebras in physics we have to develop adequate mathematical apparatus. In particular, it is necessary to have an appropriate definition of tensor operators transforming under representations of quantum groups and to prove the q-analogue of the Wigner–Eckart theorem.

Tensor operators for quantum groups and the corresponding Wigner–Eckart theorem are considered by Biedenharn and Tarlini (1990) (see also Klimyk and Smirnov 1990, Nomura 1990a, 1990b, Rittenberg and Scheunert 1991). In their definition of tensor operators, Biedenharn and Tarlini (1990) demand that action of generators E_α of a quantum algebra $U_q(g)$ upon tensor operators $\{t_n\}$ is compatible with comultiplication in $U_q(g)$. This requirement does not give a complete definition of a tensor operator as it is in the case of classical compact Lie groups. Besides, the definition by Biedenharn and Tarlini (1990) is only for tensor operators transforming under representations of quantum algebras $U_q(g)$ (q-deformed universal enveloping algebras). Although Biedenharn and Tarlini call quantum algebras $U_q(g)$ quantum groups, we differentiate between two notations: a quantum algebra $U_q(g)$ and the algebra of functions $A(G_q)$ on a quantum group G_q. These mathematical objects are dual. Nevertheless, definitions of tensor operators for $U_q(g)$ and for $A(G_q)$ are different. Of course, since $U_q(g)$ and $A(G_q)$ are dual, then these definitions are connected.

In this paper we explicitly define tensor operators transforming under a representation of the quantum group $U_q(n)$ (which is the q-analogue of the unitary group $U(n)$), that is under a corepresentation of the Hopf algebra $A(U_q(n))$, and prove the Wigner–Eckart theorem for these tensor operators (these results are absent in Biedenharn and Tarlini 1990). These tensor operators are defined in the same way as in the case of tensor operators transforming under representations of classical compact Lie groups. The duality between the Hopf algebra $A(U_q(n))$ and the Hopf algebra $U_q(u(n))$ (q-deformation of the universal enveloping algebra of the Lie algebra of the group $U(n)$) allows us to go over to definition of tensor operators transforming under

0305-4470/92/102919+09$04.50 © 1992 IOP Publishing Ltd

representations of the algebra $U_q(u(n))$ and to obtain the Wigner-Eckart theorem for them. In this way we receive the explicit formula of action of the generators E_α of the algebra $U_q(u(n))$ upon tensor operators. Thus, for the case of the quantum algebra $U_q(u(n))$ we derive the explicit expression for the left-hand side of formula (9) of the paper by Biedenharn and Tarlini (1990), such that $\{t_n\}$ satisfies the conditions of the main theorem of that paper.

2. The quantum algebras $U_q(\mathrm{gl}(n, C))$ and $U_q(u(n))$

Let L be the lattice $\frac{1}{2}\Sigma_{k=1}^{n} \boldsymbol{Z}\boldsymbol{e}_k$ in $\boldsymbol{R}^n$, where $\boldsymbol{Z}$ is the set of all integers and $\boldsymbol{e}_k$ are the unit vectors. With the help of the formula $\langle e_i, e_j\rangle = \delta_{ij}$ a scalar product is introduced in L. The quantum algebra $U_q(\mathrm{gl}(n, C))$ is generated by the elements q^a, $\alpha \in L$, e_k, f_k, $1 \leqslant k < n$, which obey the relations

$$q^0 = 1 \qquad q^a q^b = q^{a+b} \qquad a, b \in L \tag{1}$$

$$q^a e_k q^{-a} = q^{\langle a, e_k - e_{k+1}\rangle} e_k \tag{2}$$

$$q^a f_k q^{-a} = q^{-\langle a, e_k - e_{k+1}\rangle} f_k \qquad a \in L \qquad 1 \leqslant k < n \tag{3}$$

$$e_i f_j - f_j e_i = \frac{q^{e_i - e_{i+1}} - q^{-e_i + e_{i+1}}}{q - q^{-1}} \delta_{ij} \tag{4}$$

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \qquad |i - j| = 1 \tag{5}$$

$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \qquad |i - j| = 1 \tag{6}$$

$$e_i e_j = e_j e_i \qquad f_i f_j = f_j f_i \qquad |i - j| > 1. \tag{7}$$

The structure of a Hopf algebra is defined in $U_q(\mathrm{gl}(n, C))$. Namely, the coproduct Δ, the co-unit ε and the antipode S are given by the formulae (Jimbo 1986b)

$$\Delta(q^a) = q^a \otimes q^a \qquad a \in L \tag{8}$$

$$\Delta(e_k) = e_k \otimes q^{-(e_k - e_{k+1})/2} + q^{(e_k - e_{k+1})/2} \otimes e_k \tag{9}$$

$$\Delta(f_k) = f_k \otimes q^{-(e_k - e_{k+1})/2} + q^{(e_k - e_{k+1})/2} \otimes f_k \tag{10}$$

$$\varepsilon(q^a) = 1 \qquad \varepsilon(e_k) = \varepsilon(f_k) = 0 \tag{11}$$

$$S(q^a) = q^{-a} \qquad S(e_k) = -q^{-1} e_k \qquad S(f_k) = -q f_k. \tag{12}$$

By means of $*$-operations (which are antilinear antiautomorphisms) real forms of $U_q(\mathrm{gl}(n, C))$ can be separated. The compact quantum algebra $U_q(u(n))$ is defined with the help of the $*$-operation

$$(q^a)^* = q^a \qquad f_k^* = e_k \qquad e_k^* = f_k \tag{13}$$

in $U_q(\mathrm{gl}(n, C))$. It is a $*$-Hopf algebra.

Finite dimensional irreducible representations T of the algebra $U_q(\mathrm{gl}(n, C))$ [and of the algebra $U_q(u(n))$] are in one-to-one correspondence with such representations of the classical unitary group $\mathrm{U}(n)$ (Jimbo 1986a) and are given in the Gel'fand-Tsetlin basis $\{|M\rangle\}$ by the Gel'fand-Tsetlin formulae in which all factorial $m!$ are replaced by the corresponding q-factorials $[m]!$ defined by the formula

$$[m]! = [m][m-1] \ldots [1] \qquad \text{where } [k] = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

In particular, the operators $E_j = T(e_j)$, $F_j = T(f_j)$, $T(q^{e_j}) \equiv q^{H_j}$ act upon the basis vectors $|M\rangle$ by the formulae

$$E_j|M\rangle = \sum_{k=1}^{j} A_k(M)|M_j^{+k}\rangle \tag{14}$$

$$F_j|M\rangle = \sum_{k=1}^{j} A_k(M_j^{-k})|M_j^{-k}\rangle \tag{15}$$

$$H_j|M\rangle = \left(\sum_{i=1}^{j} m_{ij} - \sum_{i=1}^{j-1} m_{i,j-1}\right)|M\rangle \tag{16}$$

where $M_j^{\pm k}$ is the Gel'fand–Tsetlin pattern obtained from M by replacing m_{kj} by $m_{kj} \pm 1$ and $A_k(M)$ are coefficients which are explicitly given by Jimbo (1986a).

The tensor product of two irreducible finite dimensional representations $T(m_1)$ and $T(m_2)$ of the algebra $U_q(u(n))$ with highest weights m_1 and m_2 decomposes into the direct sum of irreducible representations:

$$T(m_1) \otimes T(m_2) = \sum_{m,r} \oplus T(m, r) \tag{17}$$

where r labels multiple irreducible representations $T(m)$ and $T(m, r) \equiv T(m)$ for all values of r. If $|N\rangle, |K\rangle, |M\rangle$ are the Gel'fand–Tsetlin bases of the carrier spaces of the irreducible representations $T(m_1)$, $T(m_2)$, $T(m)$ respectively, then according to decomposition (17) the Clebsch–Gordan coefficients of this tensor product are defined as

$$|M\rangle_r = \sum_{N,K} C^{m_1 m_2 m}_{NKMr} |N\rangle \otimes |K\rangle. \tag{18}$$

As in the classical case, we have the orthogonality relations

$$\sum_{N,K} C^{m_1 m_2 m}_{NKMr} \overline{C^{m_1 m_2 m'}_{NKM'r'}} = \delta_{mm'}\delta_{MM'}\delta_{rr'} \tag{19}$$

$$\sum_{m,M,r} C^{m_1 m_2 m}_{NKMr} \overline{C^{m_1 m_2 m}_{N'K'Mr}} = \delta_{NN'}\delta_{KK'} \tag{20}$$

where the bar denotes complex conjugation.

3. The algebra of functions on the quantum group $U_q(n)$

In the basis $\{|M\rangle\}$ the irreducible representation $T(m)$ of the quantum algebra $U_q(u(n))$ is given by the matrix with matrix elements t^m_{MN} depending on elements $a \in U_q(u(n))$. As in the case of the quantum group $SU_q(2)$ (Groza *et al* 1990), formulae (8)–(10) mean that

$$t^m_{MN} t^{m'}_{KR} \neq t^{m'}_{KR} t^m_{MN}.$$

As in the classical case, we have the relations

$$\sum_{r,m,M,L} C^{m_1 m_2 m}_{NKMr} \overline{C^{m_1 m_2 m}_{RSLr}} t^m_{ML} = t^{m_1}_{NR} t^{m_2}_{KS} \tag{21}$$

$$\sum_{N,K,R,S} C^{m_1 m_2 m}_{NKMr} \overline{C^{m_1 m_2 m}_{RSLr}} t^{m_1}_{NR} t^{m_2}_{KS} = t^m_{ML} \tag{22}$$

where $C^{\cdots}_{\cdots}$ are Clebsch–Gordan coefficients of the quantum algebra $U_q(u(n))$.

For the irreducible representation T_1 of the algebra $U_q(u(n))$ with highest weight $(1,0,\ldots,0)$ (the vector representation) elements of the Gel'fand-Tsetlin basis are labelled by the Gel'fand-Tsetlin patterns with rows of the types $(1,0,\ldots,0)$ and $(0,\ldots,0)$. We denote by v_j the element of this basis whose pattern consists of j rows of the type $(1,0,\ldots,0)$. Then the elements v_i, $i=1,2,\ldots,n$, form a basis of the carrier space of the representation T_1. Let t_{ij}, $1\leq i, j\leq n$, be the matrix elements of this representation with respect to the basis v_i, $i=1,2,\ldots,n$. Using relations (21) and (22) for $m_1=m_2=(1,0,\ldots,0)$ and explicit expressions for Clebsch-Gordan coefficients of the tensor product $T_1\otimes T_1$ (Pasquier 1988) in the same way as in the paper by Groza *et al* (1990) we derive that t_{ij}, $1\leq i, j\leq n$, satisfy the relations

$$t_{ik}t_{jk}=qt_{jk}t_{ik} \qquad t_{ki}t_{kj}=qt_{kj}t_{ki} \qquad i<j \tag{23}$$

$$t_{il}t_{jk}=t_{jk}t_{il} \qquad i<j \qquad k<l \tag{24}$$

$$t_{ik}t_{jl}-qt_{il}t_{jk}=t_{jl}t_{ik}-q^{-1}t_{jk}t_{il} \qquad i<j \qquad k<l \tag{25}$$

$$t_{ij}\det{}_q^{-1}=\det{}_q^{-1}t_{ij} \qquad \det{}_q\det{}_q^{-1}=\det{}_q^{-1}\det{}_q=1 \tag{26}$$

which are usually derived with the help of universal R matrix. Here $\det_q$ is the q-determinant defined by the formula

$$\det{}_q=\sum_{s\in S_n}(-q)^{\sigma(s)}t_{1,s(1)}\ldots t_{n,s(n)}$$

where S_n is the permutation group of the set $1,2,\ldots,n$ and $\sigma(s)$ is the number of inversions involved in s. We generate by the elements t_{ij}, $1\leq i, j\leq n$, $\det_q^{-1}$ obeying relations (23)–(26) the associative algebra. The structure of a Hopf algebra is introduced into this algebra (Noumi *et al* 1990, Parshall and Wang 1991). According to this structure the comultiplication Δ_A and the co-unit ε_A are defined by the formulae

$$\Delta_A(t_{mn})=\sum_{k=1}^{n}t_{mk}\otimes t_{kn}$$

$$\Delta_A(\det{}_q^{-1})=\det{}_q^{-1}\otimes\det{}_q^{-1}$$

$$\varepsilon_A(t_{ij})=\delta_{ij} \qquad \varepsilon_A(\det{}_q^{-1})=1.$$

The antipode S_A is uniquely determined by the relations

$$S_A(t_{ij})=(-q)^{i-j}a_{ji}\det{}_q^{-1} \qquad S_A(\det{}_q)=\det{}_q^{-1}.$$

Here a_{ji} is the corresponding q-minor

$$a_{ji}=\sum_{s\in S_{n-1}}(-q)^{\sigma(s)}t_{j_1,s(i_1)}\ldots t_{j_{n-1},s(i_{n-1})}$$

where $(j_1,\ldots,j_{n-1})$ is the set $(1,2,\ldots,n)$ without j and $(i_1,\ldots,i_{n-1})$ is the set $(1,2,\ldots,n)$ without i.

The algebra generated by the elements t_{ij}, $1\leq i, j\leq n$, $\det_q^{-1}$, obeying relations (23)–(26), with the structure of a Hopf algebra described is denoted by $A(\mathrm{GL}_q(n, C))$ and is called the algebra of functions on the quantum group $\mathrm{GL}_q(n, C)$. The matrix elements t^{m}_{MN} from formulae (21) and (22) belong to this algebra. The structure of a $*$-Hopf algebra can be defined in $A\equiv A(\mathrm{GL}_q(n, C))$. We put

$$t^*_{ij}=S(t_{ji})=(-q)^{j-i}a_{ij}\det{}_q^{-1} \qquad \det{}_q^*=\det{}_q^{-1} \tag{27}$$

and extend this $*$-operation onto all elements of A considering that $*$ is an antilinear antiautomorphism, that is

$$(\alpha a+\beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^* \qquad \alpha, \beta \in C \qquad a, b \in A$$

$$(ab)^* = b^* a^* \qquad a, b \in A.$$

The Hopf algebra A with this $*$-operation is called the algebra of functions on the quantum group $U_q(n)$ and is denoted by $A(U_q(n))$.

A matrix (t^m_{MN}), described above, with a fixed highest weight $m=(m_{1n}, \ldots, m_{nn})$ is called a corepresentation of the Hopf algebra $A(GL_q(n, C))$ and of the $*$-Hopf algebra $A(U_q(n))$. It is also called a representation (or a matrix representation) of the quantum group $GL_q(n, C)$ and of the quantum group $U_q(n)$. The detailed description of these representations can be found in Noumi *et al* (1990).

There is an invariant linear functional (invariant integral) φ on $A(U_q(n))$ which is defined by the relations (Noumi *et al* 1990)

$$[(\mathrm{id}\otimes\varphi)\circ\Delta_A](a) = I\cdot\varphi(a)$$

$$[(\varphi\otimes\mathrm{id})]\circ\Delta_A](a) = \varphi(a)\cdot I \qquad a\in A(U_q(n))$$

where I is the unit element from A and id is the identity operator on A. It is uniquely determined by the condition

$$\varphi(t^0_{00}) = \varphi(I) = 1 \qquad \varphi(t^m_{MN}) = 0 \quad \text{for all} \quad m \neq (0, \ldots, 0).$$

Using the invariant integral φ the scalar product $(a, b) = \varphi(ab^*)$ is defined in $A(U_q(n))$. It is proved (Noumi *et al* 1990) that for matrix elements t^m_{MN} [which are elements of the algebra $A(U_q(n))$] we have

$$\varphi(t^m_{MN}, t^{m'}_{PQ}) = 0 \quad \text{if} \quad (m, M, N) = (m', P, Q) \tag{28}$$

$$\varphi(t^m_{MN}, t^m_{MN}) = (d(m))^{-1} q^{-2(\rho, w(N))} \equiv c(m, N) \tag{29}$$

where $w(N)$ is the weight of the vector labelled by the Gel'fand–Tsetlin pattern N, $d(m)$ is the q-dimension of the representation $T(m)$, that is

$$d(m) = \sum_M q^{2(\rho, w(M))}$$

[the sum is over all Gel'fand–Tsetlin patterns M of the representation $T(m)$] and

$$2\rho = \sum_{k=1}^{n} (n+1-2k) e_k.$$

4. Wigner–Eckart theorem for the quantum group $U_q(n)$

Let $T = \Sigma_m \oplus T(m)$ be a representation of the quantum group $U_q(n)$ which is a direct sum of irreducible matrix representations of $U_q(n)$. For the sake of simplicity we suppose that multiplicities of representations $T(m)$ in T do not exeed 1. Let R_M, $M\in\Omega$ [where Ω is the set of Gel'fand–Tsetlin patterns for the representation $T(m')$], be a set of operators (numerical matrices) of the dimension equal to that of the representation T. We say that $\boldsymbol{R} = \{R_M, M\in\Omega\}$ is a tensor operator transforming under the representation $T(m') = (t^{m'}_{MN})$ of the quantum group $U_q(n)$ if the relations

$$TR_M T^* = \sum_{N\in\Omega} t^{m'}_{NM} R_N \tag{30}$$

are fulfilled for all $M \in \Omega$ where the equality is understood as element-wise equality of matrices and $T^* = \Sigma_m \oplus T^*(m)$ means the matrix which is the direct sum of matrices $T^*(m) = (t^m_{PQ})^* = (t^{m^*}_{QP})$. Here the $*$-operation is given by formula (27).

The equality $T^*(m)T(m) = E$ is valid, where E is the unit matrix with the units I of the algebra $A(\mathrm{U}_q(n))$ on the main diagonal. Hence, formula (30) can be written as

$$TR_M = \sum_{N \in \Omega} t^{m'}_{NM} R_N T.$$

Writing down the basis elements $|M\rangle$ of the carrier space of the representation $T(m)$ in the form $|m, M\rangle$ we can represent this formula as

$$\sum_{Q \in \Omega_m} t^m_{PQ} \langle m, Q|R_M|m'', D\rangle = \sum_{N \in \Omega} \sum_{E \in \Omega_{m''}} t^{m'}_{NM} \langle m, P|R_N|m'', E\rangle t^{m''}_{ED}. \tag{31}$$

This equality has to be understood as an equality in the algebra $A(\mathrm{U}_q(n))$.

By means of relations (28) and (29) we obtain from (31) that

$$c(m, Q)\langle m, Q|R_M|m'', D\rangle = \sum_{N \in \Omega} \sum_{E \in \Omega_{m''}} \langle m, P|R_N|m'', E\rangle (t^{m'}_{NM} t^{m''}_{ED}, t^m_{PQ}).$$

It follows from (21) that

$$(t^{m'}_{NM} t^{m''}_{ED}, t^m_{PQ}) = c(m, Q) \sum_r C^{m'm''m}_{NEPr} \overline{C^{m'm''m}_{MDQr}}.$$

Therefore,

$$\langle m, Q|R_M|m'', D\rangle = \sum_{r,N,E} \langle m, P|R_N|m'', E\rangle C^{m'm''m}_{NEPr} \overline{C^{m'm''m}_{MDQr}}.$$

As in the classical case, we obtain from here that

$$\langle m, Q|R_M|m'', D\rangle = \sum_r \langle m\|\boldsymbol{R}\|m''\rangle_r \overline{C^{m'm''m}_{MDQr}} \tag{32}$$

where $\langle m\|\boldsymbol{R}\|m''\rangle_r$ are the reduced matrix elements which do not depend on Q, M, D. The reduced matrix elements are expressed in terms of the matrix elements from the left-hand side of (32). Namely, due to orthogonality relation (19) we have

$$\langle m\|\boldsymbol{R}\|m''\rangle_r = (\dim T(m))^{-1} \sum_{M,D,Q} \langle m, Q|R_M|m'', D\rangle C^{m'm''m}_{MDQr} \tag{33}$$

where $\dim T(m)$ is the usual dimension of the representation $T(m)$. The formulae (32) and (33) are the Wigner–Eckart theorem for the quantum group $\mathrm{U}_q(n)$ or for the Hopf algebra of functions on $\mathrm{U}_q(n)$.

It follows from formula (32) that $\langle m, Q|R_M|m'', D\rangle$ may be non-vanishing only if for all k $(1 \le k \le n)$ we have

$$\sum_{i=1}^{k} m_{ik} + \sum_{i=1}^{k} d_{ik} = \sum_{i=1}^{k} q_{ik} \tag{34}$$

where m_{ik}, d_{ik}, q_{ik} are the entries of the Gel'fand–Tsetlin patterns M, D, Q. Namely, for this case the Clebsch–Gordan coefficients $C^{m'm''m}_{MDQr}$ may be non-vanishing.

5. The Wigner–Eckart theorem for the quantum algebra $U_q(u(n))$

The quantum algebra $U_q(gl(n, C))$ is dual to the Hopf algebra $A = A(\mathrm{GL}_q(n, C))$. The elements q^a, e_k, f_k of $U_q(gl(n, C))$ are considered as the linear functionals on A which act upon t_{ij} and $\det_q^{\pm 1}$ according to the formulae

$$q^a(t_{ij}) = \delta_{ij} q^{\langle a, e_i \rangle} \qquad q^a(\det_q^r) = q^{r\langle a, e_1 + \ldots + e_n \rangle} \qquad r \in \boldsymbol{Z}$$

$$e_k(t_{ij}) = \delta_{ik}\delta_{j,k+1} \qquad f_k(t_{ij}) = \delta_{i,k+1}\delta_{jk}$$

$$e_k(\det_q^r) = f_k(\det_q^r) = 0 \qquad r \in Z.$$

The action of q^a, e_k, f_k upon other elements of A (that is upon polynomials in t_{ij} and $\det_q^{-1}$) are defined with the help of the formula

$$X(ab) = (\Delta X)(a \otimes b) \qquad X \in U_q(\mathrm{gl}(n, C)) \qquad a, b \in A.$$

Therefore, for a, $b \in A$ we have

$$q^a(ab) = q^a(a) q^a(b) \tag{35}$$

$$e_k(ab) = e_k(a) q^{-(e_k - e_{k+1})/2}(b) + q^{(e_k - e_{k+1})/2}(a) e_k(b) \tag{36}$$

$$f_k(ab) = f_k(a) q^{-(e_k - e_{k+1})/2}(b) + q^{(e_k - e_{k+1})/2}(a) f_k(b). \tag{37}$$

Products of elements q^a, e_k, f_k of $U_q(\mathrm{gl}(n, C))$ in this approach are defined with the help of formula

$$X_1 X_2(a) = (X_1 \otimes X_2)(\Delta_A a) \qquad X_1, X_2 \in U_q(\mathrm{gl}(n, C)) \qquad a \in A$$

where Δ_A is the comultiplication in A. In particular,

$$(X_1 X_2)(t^m_{MN}) = \sum_K X_1(t^m_{MK}) X_2(t^m_{KN}).$$

It can be shown (Noumi *et al* 1990) that the elements q^a, e_k, f_k, just defined, satisfy relations (1)–(7). The matrices of operators of the representation $T_m = T(m)$ of the algebra $U_q(\mathrm{gl}(n, C))$ in the basis $\{|M\rangle\}$ are obtained in the following way. If $X \in U_q(\mathrm{gl}(n, C))$, then the matrix element $T_m(X)_{MN} = \langle M | T_m(X) | N \rangle$ is evaluated by the formula

$$T_m(X)_{MN} = X(t^m_{MN})$$

where (t^m_{MN}) is the corresponding irreducible corepresentation of the Hopf algebra $A(\mathrm{GL}_q(n, C))$. In particular, it follows from (36) that

$$e_k(t^m_{MN} t^{m'}_{RS}) = T_m(e_k)_{MN} (q^{-T_{m'}(e_k - e_{k+1})/2})_{RS} + (q^{T_m(e_k - e_{k+1})/2})_{MN} T_{m'}(e_k)_{RS}. \tag{38}$$

The similar formulae for f_k and q^a can be obtained from formulae (35) and (37).

Applying formula (38) to both sides of relation (31) we have

$$\sum_Q E^m_{PQ} \langle m, Q | R_M | m'', D \rangle$$

$$= \sum_{N,E} E^{m'}_{NM} \langle m, P | R_N | m'', E \rangle \delta_{ED} q^{-\alpha/2} + \sum_{N,E} q^{\beta/2} \delta_{NM} \langle m, P | R_N | m'', E \rangle E^{m''}_{ED}$$

where $E^m_{PQ} = T_m(e_k)_{PQ}$ and

$$\alpha = -\sum_{i=1}^{k+1} d_{i,k+1} - \sum_{j=1}^{k-1} d_{j,k-1} + 2 \sum_{s=1}^{k} d_{sk}$$

$$\beta = -\sum_{i=1}^{k+1} m_{i,k+1} - \sum_{j=1}^{k-1} m_{j,k-1} + 2 \sum_{s=1}^{k} m_{sk}$$

(d_{jr} and m_{jr} are the entries of the Gel-fand–Tsetlin patterns D and M respectively). This formula can be written as

$$\sum_N E^{m'}_{NM}\langle m, P|R_N|m'', D\rangle$$
$$= \sum_Q E^{m}_{PQ}\langle m, Q|R_M|m'', D\rangle q^{\alpha/2} - \sum_E q^{(\alpha+\beta)/2}\langle m, P|R_M|m'', E\rangle E^{m''}_{ED}. \qquad (39)$$

Due to formula (14), $E^{m''}_{ED}=0$ if, for entries of the Gel'fand–Tsetlin patterns E and D, we have

$$\sum_{i=1}^{k} e_{ik} - \sum_{i=1}^{k} d_{ik} \neq 1.$$

Taking into account this formula and relation (34) we can write (39) in the operator form

$$E_k R_M q^{(H_k-H_{k+1})/2} - q q^{(H_k-H_{k+1})/2} R_M E_k = \sum_N (E_k)^{m'}_{NM} R_N \qquad (40)$$

where $E_k = T(e_k)$, $H_k = T(e_k)$ and the matrix elements $(E_k)^{m'}_{NM}$ of the operator $E^{m'}_k$ are determined by formula (14), that is

$$\sum_N (E_k)^{m'}_{NM} R_N = \sum_{j=1}^{k-1} A_j(M) R(M^{+j}_{k-1}).$$

Here for simplicity we used the notation $R(N) \equiv R_N$.

Replacing e_k by f_k and repeating the above reasonings we obtain

$$F_k R_M q^{(H_k-H_{k+1})/2} - q^{-1} q^{(H_k-H_{k+1})/2} R_M F_k = \sum_{j=1}^{k-1} A_j(M^{-j}_{k-1}) R(M^{-j}_{k-1}). \qquad (41)$$

For the operators H_k we have

$$H_k R_M - R_M H_k = \left(\sum_{i=1}^{k} m_{ik} - \sum_{j=1}^{k-1} m_{j,k-1}\right) R_M. \qquad (42)$$

Thus, we now can define tensor operators transforming under an irreducible representation $T_{m'}$ of the quantum algebra $U_q(u(n))$ or of the quantum algebra $U_q(\mathrm{gl}(n, C))$ as a set of operators R_M, $M \in \Omega$, for which relations (40), (41) for $k = 1, 2, \ldots, n-1$ and relations (42) for $k = 1, 2, \ldots, n$ are fulfilled. According to main theorem of the paper by Biedenharn and Tarlini (1990) the left-hand sides of these relations give the action of the generators of the algebra $U_q(\mathrm{gl}(n, C))$ upon R_M, $M \in \Omega$, compatible with comultiplication Δ in this algebra. We have obtained formulae (40)–(42) from the results of section 4; that is, the tensor operator defined by these formulae coincides with that of section 4. Therefore, relations (32) and (33) are valid for tensor operators defined by formulae (40)–(42). These relations give the Wigner–Eckart theorem for tensor operators from this section.

6. Conclusion

Tensor operators transforming under finite dimensional irreducible representations of the quantum group $\mathrm{U}_q(n)$ are of great importance for a number of applications. In particular, as in the case of the classical unitary group $\mathrm{U}(n)$, they can be used for development of the theory of Clebsch–Gordan coefficients of $\mathrm{U}_q(n)$. We have proved the Wigner–Eckart theorem for these tensor operators. Then using duality between the algebra of functions $A(\mathrm{U}_q(n))$ on the quantum group $\mathrm{U}_q(n)$ and the q-deformed universal enveloping algebra $U_q(u(n))$ we derive formulae defining tensor operators transforming under representations of $U_q(u(n))$ and obtain the Wigner–Eckart theorem for them. In forthcoming papers these tensor operators will be used for derivation of formulae for some Clebsch–Gordan coefficients of the quantum group $\mathrm{U}_q(n)$.

References

Biedenharn L C and Tarlini M 1990 *Lett. Math. Phys.* **20** 271

Groza V A, Kachurik I I and Klimyk A U 1990 *J. Math. Phys.* **31** 2769

Jimbo M 1986a *Lecture Notes in Physics* **246** 334

—— 1986b *Lett. Math. Phys.* **11** 247

Klimyk A U and Smirnov Yu F 1990 Wigner–Eckart theorem for the quantum algebra $\mathrm{U}_q(\mathrm{sl}_2)$ and for the quantum group $\mathrm{SU}_q(2)$ *Preprint* Kiev Institute for Theoretical Physics

Nomura M 1990a *J. Phys. Soc. Japan* **59** 439

—— 1990b *J. Phys. Soc. Japan* **59** 2345

Noumi M, Yamada H and Mimachi K 1990 Finite dimensional representations of the quantum group $\mathrm{GL}_q(n, C)$ and the zonal spherical functions on $\mathrm{U}_q(n-1)\backslash\mathrm{U}_q(n)$ *Preprint* University of Tokyo

Parshall B and Wang J-P 1991 *Memoirs Amer. Math. Soc.* **439** 1

Pasquier V 1988 *Comm. Math. Phys.* **118** 335

Rittenberg V and Scheunert M 1991 Tensor operators for quantum groups and applications *Preprint* Bonn University